Possible Definitions of an 'A Priori' Granule in General Rough Set Theory

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We introduce an abstract framework for general rough set theory from a mereological perspective and consider possible concepts of 'a priori’ granules and granulation in the same. The framework is ideal for relaxing many of the relatively superfluous set-theoretic axioms and for improving the semantics of many relation based, cover-based and dialectical rough set theories. This is a relatively simplified presentation of a section in three different recent research papers by the present author.
Outline

1. Nonstandard Introduction
2. Rough Y-Systems (RYS)
3. Granules: Possible Properties
A Space of interest in General RST consists of objects, some of which are more definite than others and some others are more approximate than others.

In general we use operations on the objects to form their approximations and may be able to define the concept of definiteness in terms of formulas constructed from them.

In many situations, we may be actually able to derive this 'space of interest' from more basic objects called 'granules' possessing different properties.
Simplest Case

- **Approximation Space**: $S = \langle S, R \rangle$, where $S$ is a set and $R$ is an equivalence.
- If $A \subseteq S$, $A^l = \bigcup \{[x] ; [x] \subseteq A\}$ and $A^u = \bigcup \{[x] ; [x] \cap A \neq \emptyset\}$ are the lower and upper approximation of $A$ respectively.
- $A$ is *definite* iff $A^l = A = A^u$.
- Rough Incusion: $A \sqsubseteq B$ iff $A^l \subseteq B^l$ and $A^u \subseteq B^u$.
- Rough Equality: $A \approx B$ iff $A \sqsubseteq B$ and $B \sqsubseteq A$. 
Example

Standard Example
Granules in RST

- Classical RST: Granules are the minimal definite elements. The set of all granules forms a granulation for the semantics.
- Classical RST: Granules are precisely the equivalence classes determined by the equivalence relation (An a priori Definition)
- In most cover based and other relation based RSTs, the most appropriate concept of granules emerge after the formulation of the semantics.
- This is problematic for semantics and applications that require some specific conditions on the granules used.
- The Lesniewski ontology based mereological approach makes heavy use of membership functions in basic definitions and has related concepts of granularity - this is problematic from the point of view of foundations.
- The scope of definitions of granules in the literature have been very restricted for many reasons.
Let $S$ be a set and $\mathcal{K} = \{K_i\}_{i=1}^n$ be a collection of subsets of it such that $\bigcup \mathcal{K} = S$. If $X \subseteq S$, then consider the sets (with $K_0 = \emptyset$, $K_{n+1} = S$)

1. $X^{l_1} = \bigcup \{K_i : K_i \subseteq X, i \in \{0, 1, ..., n\}\}$
2. $X^{l_2} = \bigcup \{\bigcap (S \setminus K_i) : \bigcap (S \setminus K_i) \subseteq X, I \subseteq \{1, ..., n + 1\}\}$
3. $X^{u_1} = \bigcap \{\bigcup K_i : X, \subseteq \bigcup K_i, i \in \{1, ..., n + 1\}\}$
4. $X^{u_2} = \bigcap \{S \setminus K_i : X, \subseteq S \setminus K_i, i \in \{0, ..., n\}\}$

The pair $(X^{l_1}, X^{u_1})$ is called a $AU$-rough set by union, while $(X^{l_2}, X^{u_2})$ a $AI$-rough set by intersection. In [AM960], we show that the elements of $\mathcal{K}$ are not the best possible granules for the approximations. Obviously the elements of $\mathcal{K}$ are not 'definite' in many senses in general.
Rough $Y$-Systems (RYS)

- $S = \langle S, W, P, (l_i)_1^n, (u_i)_1^n, +, \cdot, \sim, 1 \rangle$
- $(\forall x)Pxx ; (\forall x, y)(Pxy, Pyx \rightarrow x = y)$
- For each $i, j, l_i, u_j$ are surjective functions : $S \hookrightarrow W$
- For each $i$, $(\forall x, y)(Pxy \rightarrow P(l_ix)(l_iy), P(u_ix)(u_iy))$
- For each $i$, $(\forall x)P(l_ix)x, P(x)(u_ix)), P(l_ix)(u_il_ix), P(l_iu_ix)(u_ix)$
- For each $i$, $(\forall x)(P(u_ix)(l_ix) \rightarrow x = l_ix = u_ix)$

In the definition of a RYS, it makes sense to relax the surjectivity of $u_i, l_i$. The resulting structure will be called a general RYS.
Overlap: $O_{xy}$ iff $(\exists z) P_{zx} \land P_{zy}$; Underlap: $U_{xy}$ iff $(\exists z) P_{xz} \land P_{yz}$

Proper Part: $P_{xy}$ iff $P_{xy} \land \neg P_{yx}$; Overcross: $X_{xy}$ iff $O_{xy} \land \neg P_{xy}$

Proper Overlap: $O_{xy}$ iff $X_{xy} \land X_{yx}$

Sum: $x + y = \nu z (\forall w)(O_{wz} \leftrightarrow (O_{wx} \lor O_{wy}))$

Product: $x \cdot y = \nu z (\forall w)(P_{wz} \leftrightarrow (P_{wx} \land P_{wy}))$

Difference: $x - y = \nu z (\forall w)(P_{wz} \leftrightarrow (P_{wx} \land \neg O_{wy}))$

Associativity: Assumption: $+, \cdot$ are associative
Clarifications:

- The 'parthood relation' \( P \) is intended as a general form of 'rough inclusion'.
- Interestingly many semantics of general RST do not make use of any 'rough inclusions' at all as their intent is not to describe 'roughly equivalent objects'. Example: Cattaneo’s BZ-algebras and variants.
- In classical RST, 'supplementation' in the stricter sense, \((\neg P_{xy} \rightarrow \exists z (P_{zx} \land \neg \emptyset_{zy}))\) does not hold), while the weaker version \((\neg P_{xy} \rightarrow \exists z (P_{zx} \land \neg O_{zy}))\) is trivially satisfied due to the existence of the empty object (\(\emptyset\)).
- Non-transitivity of \( P \) can be the result of adding attributes in even simpler cases.
• Handle-Door-House example situations can happen in a more general sense.

• Problem of Ontological Innocence

• Sum operation can cause 'Plural reference', but is not a major problem here as we are dealing with an abstract object.

• General RYS with transitivity of the parthood relation can be related to 'property systems' of Vakarelov and include 'information quantum relation system' (Theorem)
Granules: Possible Properties

Representability, RA \[ \forall i, (\forall x)(\exists y_1, \ldots, y_r \in G) y_1 + y_2 + \ldots + y_r = x^l_i \text{ and } (\forall x)(\exists y_1, \ldots, y_p \in G) y_1 + y_2 + \ldots + y_p = x^u_i \]

Weak RA, WRA \[ \forall i, (\forall x)(\exists y_1, \ldots, y_r \in G) t_i(y_1, y_2, \ldots, y_r) = x^l_i \text{ and } (\forall x)(\exists y_1, \ldots, y_r \in G) t_i(y_1, y_2, \ldots, y_p) = x^u_i \]

Absolute Crispness, ACG For each \( i \), \( (\forall y \in G) y^l_i = y^u_i = y \)

Weak Crispness, WCG \( \exists i, (\forall y \in G) y^l_i = y^u_i = y. \)

Mereological Atomicity, MER \( \exists i, \)
\[ (\forall y \in G)(\forall x \in S)(Pxy, x^l_i = x^u_i = x \rightarrow x = y) \]
Granules: Continued

Lower Stability, LS \( \exists i, (\forall y \in G)(\forall x \in S)(P_{yx} \rightarrow P(y)(x^l_i)) \)

Upper Stability, US \( \exists i, (\forall y \in G)(\forall x \in S)(O_{yx} \rightarrow P(y)(x^u_i)) \)

Stability, ST Shall be the same as the satisfaction of LS and US.

Absolute Stability, AS Same as the satisfaction of ST for every \( i \)

No Overlap, NO \( (\forall x, y \in G)\neg O_{xy} \)

Full Underlap, FU \( \exists i, (\forall x, y \in G)(\exists z \in S)P_{xz}, P_{yz}, z^l_i = z^u_i = z \)

Unique Underlap, UU For at least one \( i \), \( (\forall x, y \in G)(P_{xz}, P_{yz}, z^l_i = z^u_i = z, P_{xb}, P_{yb}, b^l_i = b^u_i = b \rightarrow z = b) \)
Definition

A subset $G$ of $S$ in a RYS will be said to be an *admissible set of granules* provided the properties WRA, LS and FU are satisfied by it. Using more properties we can define posets of possible granulations

- $K$ in $AUAI$ systems are admissible granulations. But these can be refined (see AM960 for details) for the same approximations. So I refer to the former as 'initial granules' and the latter are relatively *refined granules*.

- Other definitions of granules exist in the literature, but they do not refer to specific properties or are not general enough

**Conclusion**: We have axiomatically defined a Poset of granules from a mereological perspective of general rough set theory. The motivation has been in the requirements of foundational studies, applications to cover based RST and dialectical RST.
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