Intrinsic co-Heyting boundaries and information incompleteness in Rough Set Analysis

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Abstract. Probably the distinguishing concept in incomplete information analysis is that of "boundary": in fact a boundary is precisely the region that represents those doubts arising from our information gaps. In the paper it is shown that the rough set analysis adequately and elegantly grasps this notion via the algebraic features provided by co-Heyting algebras.

1 Algebraic views of Rough Set Systems

Any Rough Set Systems, that is the family of all the rough sets induced by an Approximation Space over a set U (see the definition below), can be made into several logic-algebraic structures. In [7], for instance, the attention was focused on semi-simple Nelson algebras, Heyting algebras, double Stone algebras, three-valued Lukasiewicz algebras and Chain Based Lattices. In the present paper, Rough Set Systems are analysed from the point of view of co-Heyting algebras. This new chapter in the algebraic analysis of Rough Sets does not follow from aesthetic or completeness issues, but it is a pretty immediate consequence of interpreting the basic features of co-Heyting algebras (originally introduced by C. Rauszer [8] and investigated by W. Lawvere in the context of Continuum Physics) through the lenses of incomplete information analysis. Indeed in [3] and in [4], Lawvere pointed out the role that the co-intuitionistic negation "non" (dual to the intuitionistic negation "not") plays in grasping the geometrical notion of "boundary" as well as the physical concepts of "sub-body" and "essential core of a body" and we aim at providing an outline of how and to what extent they are mirrored by the basic features of incomplete information analysis.

1.1 Indiscernibility Spaces and Rough Sets

Given a (finite) universe U, when we take into account an equivalence relation $E \subseteq U \times U$ we assume that any equivalence class collects together the elements of U that are considered *indiscernible* from some point of view: in Rough Set Analysis, typically $a, b \in U$ are indiscernible if they share all the possible (observational) properties we are provided by some Information System (see [6]). In this perspective, any equivalence class modulo E will be called a *basic category* and the ordered pair (U, E) will be called an *indiscernibility space*. Let $\mathbf{AS}(U)$ be the atomic

Boolean algebra having U/E as set of atoms. Then $(U, \mathbf{AS}(U))$ is a 0-dimensional topological space. This space is called an *Approximation Space*, but with the same term we will also refer to the topology $\mathbf{AS}(U)$ itself. As a topology, $\mathbf{AS}(U)$ will induce a closure operator $\mathcal{C}$ and an interior operator $\mathcal{I}$ from $\wp(U)$ to $\mathbf{AS}(U)$.
In Rough Set Analysis, $\forall G \in U, \mathcal{C}(G)$ is called the *upper approximation* of G (with respect to E) and it is usually denoted by $(uE)(G)$, while $\mathcal{I}(G)$ is called the *lower approximation* of G and it is denoted by $(lE)(G)$. If $(lE)(G) = \emptyset$ (if $(uE)(G) = U$, then G is said to be *internally undefinable* (*externally undefinable*).
If two sets $G, G' \subseteq U$ are such that $(uE)(G) = (uE)(G')$ and $(lE)(G) = (lE)(G')$, then G and G' are said to be *rough equal*, $G \approx G'$. Thus *rough equality* is an equivalence relation on $\wp(U)$. Any equivalence class of subsets of $\wp(U)$ modulo the rough equality relation is called a *rough set*.

1.2 Basic algebraic concepts

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded distributive lattice.

Definition 1. The RIGHT ADJOINT to MEET: given $a, b \in L$, set $\forall x \in L$,

$$\text{(i) } a \wedge x \leq b \text{ if and only if } x \leq a \supset b; \text{ (ii) } \div a = a \supset 0.$$

Then $a \supset b$ is called the *pseudo-complement of a relative to b* and by definition it is the largest element x of L such that $a \wedge x$ is less than or equal to b, while the element $\div a$ is called the *pseudo-complement* (or *intuitionistic negation*) of a and it is the largest element x of L such that $a \wedge x = 0$.

Definition 2. HEYTING ALGEBRAS: if for any $a, b \in L$, the operation $a \supset b$ is defined, then $\mathbf{H} = (L, \vee, \wedge, \supset, \div, 0, 1)$ is called a Heyting algebra.

Definition 3. The LEFT ADJOINT to JOIN: given $a, b \in L$, set $\forall x \in L$,

$$\text{(i) } a \vee x \geq b \text{ if and only if } x \geq a \subset b; \text{ (ii) } \neg a = a \subset 1.$$

Then $a \subset b$ is called the *dual pseudo-complement of a relative to b*, and it is the smallest element x of L such that $a \vee x$ is greater than or equal to b, while $\neg a$ is called the *dual pseudo-complement* (or *co-intuitionistic negation*) of a and it is the smallest element x of L such that $a \vee x = 1$.

Definition 4. CO-HEYTING ALGEBRAS: if for any $a, b \in L$, $a \subset b$ is defined, then $\mathbf{CH} = (L, \vee, \wedge, \subset, \neg, 0, 1)$ is called a co-Heyting algebra.

Definition 5. A BI-HEYTING ALGEBRA is a bounded distributive lattice that is both a Heyting and a co-Heyting algebra.

EXAMPLES

1. The system of all closed subsets of a topological space is a co-Heyting algebra. In fact in this case given two closed subsets X, Y, $X \subset Y = \mathcal{C}(Y \cap -X)$. It follows that $\neg X = \mathcal{C}(-X) = -\mathcal{I}(X)$.
2. Any finite Heyting algebra is a bi-Heyting algebra.

We must now remark that being a bi-Heyting algebra is an interesting condition *in se* but not *per se*: indeed if we consider a finite Heyting algebra $\mathbf{H}$ such that the top element, 1, is co-prime (that is, for any $X \subseteq \mathbf{H}$ such that $1 = \bigvee X, 1 \in X$) then $\mathbf{H}$ is surely a bi-Heyting algebra, but for any $a \in \mathbf{H}$, $\neg a = 1$ if $a \neq 1$, $\neg a = 0$ otherwise: the co-intuitionistic negation carries a poor information, in this case. Therefore, though one can "a priori" say that any Rough Set System is bi-Heyting algebra (since it is a finite distributive lattice, thus a finite Heyting algebra), nevertheless this framework becomes interesting if we are able to identify the bi-Heyting algebra operations and to relate them to the features of Rough Set Analysis. So let us start with the operations that we can define on rough sets.
We have seen that any rough set Z is an equivalence class of subsets of U modulo the rough equality relation $\approx$. Thus $Z \in \wp(\wp(U))$. But since Z is uniquely determined by the lower and the upper approximation of any of its element, we can denote it by $< \mathcal{C}(Z_i), \mathcal{I}(Z_i) >$ for $Z_i \in Z$. Therefore we introduce the following map:

Definition 6. $rs : \wp(U) \longmapsto \mathbf{AS}(U) \times \mathbf{AS}(U); rs(X) =< \mathcal{C}(X), \mathcal{I}(X) >$.

Then, we have: $[X]_{\approx} = rs^{-1}(rs(X)), \forall X \subseteq U$. If the pre-image of rs is a singleton $\{X\}$, then we will denote it directly by X.
We denote the image of $\wp(U)$ along rs, by $RS(U)$ and we call it the *Rough Set System* induced by $\mathbf{AS}(U)$. Clearly, if $< X, Y > \in RS(U)$, then $X \supseteq Y$. Particularly, if $G \in \mathbf{AS}(U)$ then $\mathcal{C}(G) = \mathcal{I}(G)$; hence any element of $rs(\mathbf{AS}(U))$ has the form $< X, X >$. We will call any rough set of this kind an *exact rough set*. Then, let us consider the following families, for any Boolean algebra $\mathbf{B}$:
$\mathcal{P}(\mathbf{B}) = \{< a_1, a_2 > \in \mathbf{B}^2 : a_1 \geq a_2\}$; $\mathcal{B}(\mathbf{B}) = \{< a_1, a_2 > \in \mathbf{B}^2 : a_1 = a_2\}$.
Clearly $rs(\mathbf{AS}(U)) = \mathcal{B}(\mathbf{AS}(U)) \subseteq RS(U) \subseteq \mathcal{P}(\mathbf{AS}(U))$.
Before short, we will identify precisely $RS(U)$.
From now on by $a =< a_1, a_2 >, x =< x_1, x_2 >$ and so on, we will denote elements of $\mathcal{P}(\mathbf{B})$.

BASIC OPERATIONS ON $\mathcal{P}(\mathbf{B})$
1. $1 =< 1, 1 >$;
2. $a \vee b =< a_1 \vee b_1, a_2 \vee b_2 >$;
3. $a \longrightarrow b =< a_2 \longrightarrow b_1, a_2 \longrightarrow b_2 >$ (weak implication);
4. $\sim a =< \neg a_2, \neg a_1 >$ (strong negation);
where $1, \vee, \longrightarrow$ and $\neg$ applied inside the ordered pairs are the operations of the

underlying Boolean algebra **B**.

From the above basic set, we can define other derived operations. Some of these are the operations that make it possible to made $\mathcal{P}(\mathbf{B})$ into a Heyting algebra and a co-Heyting algebra.

DERIVED OPERATIONS ON $\mathcal{P}(\mathbf{B})$
5. $0 = \sim 1 = < 0, 0 >$;
6. $a \wedge b = \sim (\sim a \vee \sim b) = < a_1 \wedge b_1, a_2 \wedge b_2 >$;
7. $a \supset b = \sim \neg \sim a \vee b \vee (\neg a \wedge \neg \sim b)$ (intuitionistic implication);
8. $\div a = \sim \neg \sim a$ (intuitionistic negation);
9. $a \subset b = \sim (\sim a \supset \sim b) = \neg a \wedge b \wedge (\sim \neg \sim a \vee \sim \neg b)$;
10. $\neg a = a \longrightarrow 0 = < \neg a_2, \neg a_2 >$ (weak negation),
where $0, \wedge$ and $\neg$ applied inside the ordered pairs are the operations of **B**.

Lemma 7. *For any $a \in \mathcal{P}(\mathbf{B})$, $\div a = a \supset 0$.*

Proposition 8. FACTS ABOUT NEGATIONS*: For any $a \in \mathcal{P}(\mathbf{B})$*
1. a) $\sim\sim a = a$; b) $\neg\neg\neg a = \neg a$; c) $\div \div \div a = \div a$.
2. a) $\div \div a = \neg \div a = \sim \div a = \neg \sim a$; b) $\neg\neg a = \div \neg a = \div \sim a = \sim \neg a$;
3. *If $a = < x, x >$, then $\sim a = \neg a = \div a$.*

Proposition 9. *For any Boolean algebra **B**,*
1. $\mathbf{N}(\mathbf{B}) = (\mathcal{P}(\mathbf{B}), \wedge, \vee, \longrightarrow, \sim, \neg, 0, 1)$ *is a semi-simple Nelson algebra.*
2. $\mathbf{H}(\mathbf{B}) = (\mathcal{P}(\mathbf{B}), \wedge, \vee, \supset, \div, 0, 1)$ *is a Heyting algebra.*

(The proofs for the above Lemma, Facts and Propositions are in [7]).

Proposition 10. *For any Boolean algebra **B**, $\mathbf{CH}(\mathbf{B}) = (\mathcal{P}(\mathbf{B}), \wedge, \vee, \subset, \neg, 0, 1)$ is a co-Heyting algebra.*

Proof For any $a, b \in \mathcal{P}(\mathbf{B})$, $a \leq b$ iff $\sim b \leq \sim a$ and $\sim (a \wedge b) = \sim a \vee \sim b$: these properties of $\sim$ are inherited from the same properties of the negation of **B**. Thus, since from Proposition 9, $a \supset b = \bigvee\{z \in \mathcal{P}(\mathbf{B}) : z \wedge a \leq b\}$, it follows that $\sim (\sim a \supset \sim b) = \sim \bigvee\{z \in \mathcal{P}(\mathbf{B}) : z \wedge \sim a \leq \sim b\} = \sim \bigvee\{z \in \mathcal{P}(\mathbf{B}) : \sim\sim b \leq \sim (z \wedge \sim a)\} = \sim \bigvee\{z \in \mathcal{P}(\mathbf{B}) : b \leq \sim z \vee a\} = \bigwedge\{\sim z \in \mathcal{P}(\mathbf{B}) : b \leq \sim z \vee a\}$. Since $\sim$ is an involutive anti-isomorphism we obtain that $\subset$ is a dual relative pseudocomplementation.
Moreover, $a \subset 1 = \neg a \wedge 1 \wedge (\sim \neg \sim a \vee \sim \neg 1) = \neg a \wedge (\sim \neg \sim a \vee 1) = \neg a$. QED

Corollary 11. *For any Boolean algebra **B**, $\mathbf{BH}(\mathbf{B}) = (\mathcal{P}(\mathbf{B}), \wedge, \vee, \subset, \neg, \supset, \div, 0, 1)$ is a bi-Heyting algebra.*

2 Rough Sets and bi-intuitionistic operations

Proposition 12. $\forall X \subseteq U$,
1) $\sim (rs(X)) = rs(-X)$; 2) $\div(rs(X)) = rs(-\mathcal{C}(X))$; 3) $\neg(rs(X)) = rs(-\mathcal{I}(X))$.

Proof: 1) $\sim (rs(X)) = \sim < \mathcal{C}(X), \mathcal{I}(X) > = < -\mathcal{I}(X), -\mathcal{C}(X) > = < \mathcal{C}(-X), \mathcal{I}(-X) > = rs(-X)$; 2) $\div(rs(X)) = \div < \mathcal{C}(X), \mathcal{I}(X) > = < -\mathcal{C}(X), -\mathcal{C}(X) > = rs(-\mathcal{C}(X))$;
3) $\neg(rs(X)) = \div < \mathcal{C}(X), \mathcal{I}(X) > = < -\mathcal{I}(X), -\mathcal{I}(X) > = rs(-\mathcal{I}(X))$. QED
Combining 1 with 2 and 1 with 3 we obtain:

Corollary 13. *For any Rough Set System $RS(U)$, for any $X \subseteq U$,*
1) $\neg \sim (rs(X)) = rs(\mathcal{C}(X))$; 2) $\sim \neg(rs(X)) = rs(\mathcal{I}(X))$.
3) $\sim \neg(RS(U)) = \neg \sim (RS(U)) = rs(\mathbf{AS}(U)) = \neg(RS(U)) = \div(RS(U))$.

Proof: 1) $\neg \sim (rs(X)) = rs(- - \mathcal{C}(X))$; 2) $\sim \neg(rs(X)) = rs(- - \mathcal{I}(X))$. 3) Since both $\mathcal{C}$ and $\mathcal{I}$ map $\wp(U)$ onto $\mathbf{AS}(U)$. QED
It follows that $\neg \sim$ (that is $\div\div$) and $\sim \neg$ (that is $\neg\neg$) can be considered as modal operators that parallel with (uE) and, respectively, (lE).

3 Bodies and Boundaries via Co-Heyting algebras

Given an element a of a co-Heyting algebra **CH**, Lawvere calls $\neg\neg a$ the *regular core* of a. Generally $\neg\neg a \leq a$. In order to appreciate this term, consider the above results: if $a = rs(X)$ for $X \subseteq U$, then $rs^{-1}(\neg\neg a) = (lE)(X)$, that is the *necessary* part of X, (in a literal sense when $\mathbf{AS}(U)$ is interpreted as an S5 modal space -see for instance [5]). Moreover in [3], it is claimed that a part a may be considered a *sub-body* (or shortly a *body*) if and only if $\neg\neg a = a$. Thus the notion of "sub-body" coincides in Rough Set Systems with that of *exact rough set*, that is rough sets "(deductively) closed" and "perfect": and not by chance we are following the terminology used by Leibniz for describing the notion of "individual substance" (*Discourse on Metaphysics*).
But everything is centred on the fact that in co-Heyting algebras we can recapture the geometrical notion of "boundary". Indeed Lawvere points out that this notion is definable by means of the co-intuitionistic negation, $\neg$, in the following manner (for a belonging to any co-Heyting algebra):

$$\partial(a) = a \wedge \neg a.$$

First of all, $\partial(a)$ is the boundary of a in a topological sense: if the given co-Heyting algebra is that of Example 1, then a is a closed set of some topological space. Thus $a \cap \neg(a) = \mathcal{C}(a) \cap -\mathcal{I}(a)$, that is exactly the topological boundary, $\mathcal{F}(a)$, of a.
More generally, $\partial(a)$ is a boundary since for any a, b of any co-Heyting algebra, it formally fulfils the rules:

1) $\partial(a \wedge b) = (\partial(a) \wedge b) \vee (a \wedge \partial(b))$; 2) $\partial(a \wedge b) \vee \partial(a \vee b) = \partial(a) \vee \partial(b)$.
The first formula is called "Leibniz formula" by W. Lawvere who underlines that though its validity for boundaries of closed sets is supported by our space intuition (think of two partially overlapping ovals), nevertheless it is virtually unknown in general topology literature.
Indeed we can notice that it is essentially the usual Leibniz rule for differentiation of a product (but see also the Grassmann rule).
Moreover, Lawvere notices that any element a in a co-Heyting algebra is the join of its core and its boundary: $a = \neg\neg a \vee \partial(a)$.
All these relations apply to any rough set. Particularly, if $a = rs(X)$, then $\partial(a) = a \wedge \neg a = < a_1 \cap -a_2, a_2 \cap -a_2 > = < \mathcal{C}(X) \cap -\mathcal{I}(X), \emptyset > = < \mathcal{F}(X), \emptyset >$ (since the interior of $\partial(a)$ is always empty, one can deduce the two-dimensionality of the rough sets with non empty interior, that is rough sets corresponding to internally definable subsets of the Approximation Space). Moreover, the boundary of X in the topological space $(U, \mathbf{AS}(U))$ is given by: $\mathcal{F}(X) = rs^{-1}(\neg \sim (\partial(a)))$. In fact if $Z \in rs^{-1}(\partial(a))$, then $\mathcal{C}(Z) = \mathcal{F}(X)$ and the preceding equation follows from Corollary 13.
Now let us notice, that no boundary can contain *isolated* points of the given topology. This means that if a basic category $A \in U/E$ is a singleton, say $\{y\}$, then for no $X \subseteq U$, $A \subseteq \mathcal{F}(X)$. In fact, either $y \in X$ or not. But this means $A \subseteq \mathcal{I}(X)$ or $A \subseteq -\mathcal{C}(X)$: *tertium non datur*, whereas any non-singleton basic category M, say $\{y, z\}$, a third possibility is allowed. In fact, we can find at least two sets Y, Z such that $y \in Y, z \notin Y$ and $z \in Z, y \notin Z$ (for instance we can trivially consider $Y = \{y\}$ and $Z = \{z\}$). In this case neither $M \subseteq \mathcal{I}(Y)$ nor $M \subseteq -\mathcal{C}(Y)$, (the same for Z).
But in Rough Set Analysis x is an isolated point if and only if we have a *complete information* about it: the basic category including x reduces to a singleton and this means that x can be discerned from any other object. Let us denote by B the union of all the isolated points of $(U, \mathbf{AS}(U))$, and let us set $P = U \cap -B$. Since $a \vee \div a = \sim \partial(a)$ and $\sim < P, \emptyset > = < U, B >$, we have just established that:

Proposition 14. *For any Approximation Space* $\mathbf{AS}(U)$*, for any* $a \in \mathcal{P}(\mathbf{AS}(U))$*, the following are equivalent:*
1. $\exists X \subseteq U$ *such that* $a = rs(X)$*;*
2. $a_1 \cap B = B \cap a_2$*;*
3. $a \vee \div a \geq < U, B >$*;*
4. $\partial(a) \leq < P, \emptyset >$.

It is worth noticing that condition 14.3 claims that the *tertium non datur* $a \vee \div a$ must hold with respect to the *local top* $< U, B >$. Dually, 14.4 says that the *contradiction* $a \wedge \neg a$ must be invalid with respect to the *local bottom* $< P, \emptyset >$.
Now the algebraic task is to find an appropriate filter in order to distinguish $RS(U)$ within $\mathcal{P}(\mathbf{AS}(U))$. In and [7] a suitable filter for condition 14.2 has been introduced. Now we show that it makes the other new constraints work as well.
Indeed in the quoted papers, it is proved that $RS(U)$ can be recovered from $\mathcal{P}(\mathbf{AS}(U))$ via the filter generated by $< U, P >$. Shortly, let us set for $a, b \in$

$\mathcal{P}(\mathbf{AS}(U))$: a) $a \equiv b$ iff $\exists x \geq < U, P >$ s. t. $a \wedge x = x \wedge b$; b) $J(a) = \bigvee [a]_{\equiv}$.
Then we have:

Lemma 15. *For any $a \in \mathcal{P}(\mathbf{AS}(U))$, if $a = J(a)$, then $a_1 \cap B = B \cap a_2$.*

Corollary 16. $J^{<U,P>}(\mathcal{P}(\mathbf{AS}(U))) = RS(U)$.

Now we will prove:

Proposition 17. *For any $a \in \mathcal{P}(\mathbf{AS}(U))$, if $a = J(a)$ then*
1) $a \vee \div a \geq < U, B >$; *2) $a \wedge \neg a \leq < P, \emptyset >$.*

Proof: $a \vee \div a = < a_1, a_2 > \vee < \neg a_1, \neg a_1 > = < U, a_2 \cup \neg a_1 >$. But since a is a fixpoint of the operator J, in view of the previous Lemma we have $a_2 = a_2 \cup (B \cap a_1)$. Hence $a_2 \geq (B \cap a_1)$. It follows that $a_2 \cup \neg a_1 \geq (a_1 \cap B) \cup \neg a_1 = \neg a_1 \cup B \geq B$.
The proof of the second statement is obtained by duality. QED

4 Bi-Heyting algebras, Modalities and Rough Sets

We have seen that the two double negation sequences $\neg \sim$ and $\sim \neg$ can be considered a couple of modality operators. Indeed, according to the physical interpretation, in view of Corollary 13(3), given any $a \in RS(U)$, $\neg \sim a$ sends a to the smallest "sub-body" that contains a, whereas $\sim \neg a$ sends a to the largest sub-body contained in a. Since $\forall a \in \mathcal{P}(\mathbf{B}), \neg \sim a = \neg \div a$ and $\sim \neg a = \div \neg a$, we can connect our results to some interesting general properties of bi-Heyting algebras. Namely, let us define two operators $\Box_n$ and $\Diamond_n$ as follows:

Definition 18. Given a $\sigma-$complete bi-Heyting algebra $\mathbf{BH}$, $\forall a \in \mathbf{BH}$,
(i) $\Box_0 = \Diamond_0 = Id$; (ii): $\Box_{n+1} = \div \neg \Box_n$, $\Diamond_{n+1} = \neg \div \Diamond_n$;
(iii) $\Box(a) = \bigwedge_{i=1}^{n} \Box_i(a)$; (iv) $\Diamond(a) = \bigvee_{i=1}^{n} \Diamond_i(a)$.

Then in [9] it is shown that for any $a, \Box(a)$ is the largest complemented element of $\mathbf{BH}$ below a, while $\Diamond(a)$ is the smallest complemented element above a (the interested reader must take great care of the different notations: in the quoted paper the intuitionistic negation $\div$ is denoted by $\neg$ and the co-intuitionistic negation $\neg$ is denoted by $\sim$. So in that paper $\Box_1 = \neg \sim$ and $\Diamond_1 = \sim \neg$); it follows that in Rough Set Systems $\Box = \Box_1$ and $\Diamond = \Diamond_1$. In other words both the sequences $\Box_1, \Box_2, ...$ and $\Diamond_1, \Diamond_2, ...$ stabilize at step 1. These facts, as pointed out by in [9], are related to the De Morgan laws that, generally, fail in Heyting and, respectively, in co-Heyting algebras:

Definition 19. Let **H** and **CH** be a Heyting and, respectively, a co-Heyting algebra. Then: 1) **H** satisfies the De Morgan's law for $\div$, if $\forall x, y$, $\div(x \wedge y) = \div x \vee \div y$; 2) **CH** satisfies the De Morgan's law for $\neg$, if $\neg(x \vee y) = \neg x \wedge \neg y$.

One can show that in bi-Heyting algebras the law for $\div$ implies $\Box(a) = \div\neg a$ and that the law for $\neg$ implies $\Diamond(a) = \neg \div a$. The reverse of the implications does not hold, but one can prove that both the laws actually hold in Rough Set Systems:

Proposition 20. *In any rough set system $RS(U)$ the two laws of Definition 19 hold.*

Proof: In a Heyting algebra **H** the De Morgan law for $\div$ is equivalent to the fact that $\div\div(\mathbf{H})$ is a sublattice of **H** (see [2]). Dually for $\neg$ in co-Heyting algebras. But this is precisely the case for $RS(U)$: from Proposition 8, $\div\neg a = \neg\neg a$ and $\neg \div a = \div \div a$, any a, but from Corollary 13 we have immediately that $\neg\neg$ and $\div\div$ are both multiplicative and additive ($\mathcal{C}$ is additive in any topological space and $\div\div$ is multiplicative in any Heyting algebra. Dually for $\neg\neg$ and $\mathcal{I}$. -cf. also [1]-). QED

Thus in Rough Set Systems the De Morgan rules for $\div$ and $\neg$ hold. Since this is also equivalent to the fact that $\forall a, \neg a$ and $\div a$ are complemented, from Corollary 13(3) we obtain that for any exact rough set a, $\partial a = 0$. Particularly, since $\neg\neg(RS(U)) = rs(\mathbf{AS}(U)) = \mathcal{B}(\mathbf{AS}(U))$, it follows that Rough Set Systems are examples of those "lucky" situations in which sub-bodies form a Boolean algebra.

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